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THE POSSIBILITY OF EQUILIBRIUM POSITIONS OF RAPIDLY ROTATING BODIES IN GRAVITATIONAL FIELDS†

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The possibility of the equilibrium of a system of two rapidly rotating discs when their Newtonian attraction is exactly balanced by gravimagnetic repulsion is demonstrated. The dependence of the distance between the discs on the moments of momentum of the discs about the axis of symmetry is obtained. The gravitational fields of the discs are modelled using the supercritical Kerr solutions in the weak-field approximation. The condition for there to be no conical points on the axis between the discs and the condition for there to be no closed timelike lines in the General Theory of Relativity give, in the Newtonian limit, a system of two discs in equilibrium with a special distribution of the moment of momentum and the density.

In steady gravitational fields, the force acting on a test mass can be split into two parts, one of which, for low particle velocities in the weak-field approximation, is identical with the Newtonian force, while the other is a gravimagnetic (or gravitomagnetic) force. The gravimagnetic force acts in the same way as the Coriolis force and causes a precession of gyroscopes in the field of rotating masses [1, 2]; the analogy with a magnetic field which causes the Larmor rotation of electrons was discussed in [3]. Various methods of detecting the gravimagnetic force experimentally have been proposed.

In this paper we use the weak-field approximation for the steady solutions of Einstein's equations without assuming the velocities to be small compared with the velocity of light. An expression is obtained for the interaction force between rapidly rotating objects at considerable distances (Section 1). It is shown that equilibrium is only possible in a system of coaxially rotating discs if the distance between them does not exceed their dimensions. It is shown that the distribution of material in the disc must be compact since equilibrium is not possible between dust-like rings (Section 2). The central result of this paper is a proof that equilibrium is possible between two Kerr discs when their attraction is globally balanced by gravimagnetic repulsion (Section 3).

Note that it was suggested in [4] that, within the framework of the General Theory of Relativity, steady configurations of soliton solutions without supports are possible. A more careful investigation of this problem was carried out in [5–7] using the formalism of Belinskii and Zakharov [8] and it was shown that it is impossible to satisfy both conditions of regularity on the axis of symmetry between black holes: there are no conical points and no closed timelike lines.

These algebraic conditions were investigated in [9] using analytic continuation of the parameters of the solution for two Kerr black holes in the supercritical region, and the distance between them was calculated approximately for two equal discs. A direct approach, not using the procedure of analytic continuation, was developed in [10]. By a numerical investigation of the corresponding algebraic system it was found that global equilibrium is possible in a disc–black hole system, as well as merging of the discs when they approach each other and the formation of an extremal Kerr black hole and other effects. The approach developed below agrees asymptotically with the weak-field approximation for the exact solution [10].

1. We shall take the equations of the timelike geodesic and steady gravitational fields in the form

$$\frac{d}{ds}\frac{\gamma\upsilon_\alpha}{c}=-\frac{\gamma^2}{2}\left(\frac{h_{,\alpha}}{h}-\gamma_{\beta\gamma,\alpha}\upsilon^\beta\upsilon^\gamma\right)+\frac{\gamma^2\upsilon^\beta\sqrt{h}}{c}(g_{\beta,\alpha}-g_{\alpha,\beta}) \qquad (1.1)$$

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We have used the following notation here [1]: $g_\alpha \equiv -g_{0\alpha}/g_{00}$, $h \equiv g_{00}$, $\gamma_{\beta\delta} \equiv -g_{\beta\delta} + g_{\beta 0}g_{\delta 0}/g_{00}$ (α, β, δ = 1, 2, 3), $ds^2 = g_{ij}dx^i dx^j$ (i, j = 0, 1, 2, 3) is the square of the interval of steady space–time, $\gamma \equiv (1 - \upsilon^2/c^2)^{-1/2}$, υ^β are the components of the three-dimensional velocity, and the indices are juggled using the metric $\gamma_{\alpha\beta}$. For weak steady fields $g_{00} \approx 1 + 2\phi/c^2$ and motions, for which in Cartesian coordinates $\gamma_{\beta\gamma,\alpha}\upsilon^\beta\upsilon^\gamma \approx -\phi\upsilon^2/c^2$, Eqs (1.1) take the form

$$d\gamma\upsilon_a / d\tau = -\gamma^2\phi_{,\alpha}(1+\upsilon^2/c^2)+\gamma^2(\mathbf{v}\times\mathbf{H})_\alpha / c \tag{1.2}$$

Here τ is the proper time of a particle and $\mathbf{H} \equiv \operatorname{rot} \mathbf{A}$, $A_\alpha \equiv c^2 g_\alpha$. The equations of motion of a continuous medium with energy–momentum tensor T_{ij} in weak steady fields can be written in the form

$$T^i_{\alpha,i} = -\phi_{,\alpha}(2T^0_0 - T^i_i)/c^2 + (\mathbf{j}\times\mathbf{H})_\alpha / c, \quad cj_\alpha \equiv T^0_\alpha$$

For the fields occurring in Eq. (1.2) we have, from the linearized Einstein equations

$$\Delta\phi = G(2T^0_0 - T^i_i)/c^2 \tag{1.3}$$

where T^i_i is the trace of the energy–momentum tensor of the field sources, and

$$\operatorname{rot}\mathbf{H} = 16\pi G\mathbf{j}/c \tag{1.4}$$

The vector $c\mathbf{j}$ has its own components of the component T^0_α of the energy–momentum tensor of the sources. If the dimensions of the system of bodies are small compared with the scale of variation of the gravitational field and its total momentum is zero

$$\int_V T^0_\alpha dV = \sum_\alpha m_\alpha\gamma^2_\alpha\upsilon_{\alpha\alpha} = 0$$

the force acting on this system can be represented in the form

$$\nabla(M_1\phi + \mathbf{K}_1\cdot\mathbf{H}/(2c))$$

where $\mathbf{K}_1$ is the total moment of momentum of the system

$$\mathbf{K}_1 = \int_V (\mathbf{r}-\mathbf{R}_1)\times\mathbf{j}dV = \sum_\alpha m_\alpha\gamma^2_\alpha(\mathbf{r}_\alpha - \mathbf{R}_1)\times\boldsymbol{\upsilon}_\alpha$$

and M is the total mass of the system

$$c^{-2}\int_V (2T^0_0 - T^i_i)dV = \sum_\alpha m_\alpha\gamma^2_\alpha(1+\upsilon^2_\alpha/c^2)$$

If we use the approximate solutions of Laplace's equation far from the sources with mass M_2 and moment of momentum $\mathbf{K}_2$, we have

$$\phi \approx -\frac{M_2}{|\mathbf{r}-\mathbf{R}_2|}, \quad \mathbf{H} \approx -\frac{2G}{c}\nabla\left((\mathbf{K}_2\nabla)\frac{1}{|\mathbf{r}-\mathbf{R}_2|}\right)$$

In this case the assumption $\gamma_{\beta\gamma,\alpha}\upsilon^\beta\upsilon^\gamma \approx -\phi_{,\alpha}\upsilon^2/c^2$ follows from the fact that, far from the sources, in the first approximation we have $h^\alpha_\beta = -\delta^\alpha_\beta h^0_0$. Hence, the force with which one of the distant sources at rest acts on the other (body 2 on body 1, say) is

$$\mathbf{F} = \frac{G}{c^2}\nabla_1\left(-\frac{M_1 M_2 c^2}{|\mathbf{R}_1-\mathbf{R}_2|} + K_{1\alpha}K_{2\beta}\frac{\partial^2}{\partial x^\alpha_1 \partial x^\beta_2}\frac{1}{|\mathbf{R}_1-\mathbf{R}_2|}\right) \tag{1.5}$$

2. However, the interaction between extended objects is not described by such a simple formula. We will first calculate the projection on the axis of symmetry of the gravimagnetic force of interaction between two rotating rings F_{gm}. It is equal to $K_1 H_r/c$, where H_r is the radial component of the gravimagnetic field strength produced at points of the first ring by the second ring and K_A is the angular momentum of the ring A ($A = 1, 2$) about the axis of rotation. We obtain from (1.4)

$$H_r = -\frac{4GK_2L}{\pi c(L^2+R_1^2+R_2^2)^{3/2}}\int_0^\pi \frac{\cos\phi\, d\phi}{(1-\kappa\cos\phi)^{3/2}}, \quad \kappa \equiv \frac{2R_1R_2}{L^2+R_1^2+R_2^2}$$

At considerable distances $\kappa \ll 1$, and the gravimagnetic force of repulsion will be equal to $6GK_1K_2/(c^2L^4)$, in agreement with (1.5). For the force of attraction of the rings we have

$$F_{ge} = -\frac{GM_1M_2L}{\pi(L^2+R_1^2+R_2^2)^{3/2}}\int_0^\pi \frac{d\phi}{(1-\kappa\cos\phi)^{3/2}}$$

Hence, the total force acting on ring 1 will be equal to

$$F_{ge}\left(1-\frac{4K_1K_2}{M_1M_2c^2}f(\kappa)\right)$$

A graph of the function $f(\kappa)$ is shown in Fig. 1. If we assume that the material of the rings consists of dust, we have $K_A = 2\pi\rho_A R_A\gamma_A^2$, where ρ_A is the scalar mass per unit length of the ring A, and $M_A = 2\pi\rho_A R_A\gamma_A^2(1+\upsilon_A^2/c^2)$ ($A = 1, 2$). The total force acting on ring 1 can then be represented in the form

$$F_{ge}\left(1-\frac{2\upsilon_1/c}{1+\upsilon_1^2/c^2}\,\frac{2\upsilon_2/c}{1+\upsilon_2^2/c^2}f(\kappa)\right)$$

The expression in parentheses is always greater than zero and vanishes provided $\upsilon_A \to c$ ($A = 1, 2$), $\kappa \to 1$. Consequently, equilibrium between rings consisting of dust matter, in the weak-field approximation, turns out to be impossible for any rotational velocities less than the velocity of light and the corresponding exact solution in the General Theory of Relativity (if it exists) will not have a limit in the weak-field approximation. Hence, the distribution of matter in discs which are in equilibrium must be fairly compact.

We will now consider the forces of interaction between two extended discs close to one another when the dimensions of the discs are much greater than the distance between them. Suppose the density of the material per unit area in the discs is σ_A and the momentum density is $\mathbf{i}_A$. These quantities are obtained by integrating the components of the energy–momentum tensor $2T_0^0 - T_i^i$ and T_α^0/c over the small thickness of the disc. Then, in the region of disc 2 in view of the assumption that the distance between the discs is much less than the dimensions of the discs, we have that asymptotically $\phi_{2,z} \approx -2\pi\sigma_1$, $\mathbf{n}\times\mathbf{H}_2 \approx -8\pi\mathbf{i}_2/c$. Hence, the force acting on unit area of the first disc in the region inside it (ignoring edge effects) is

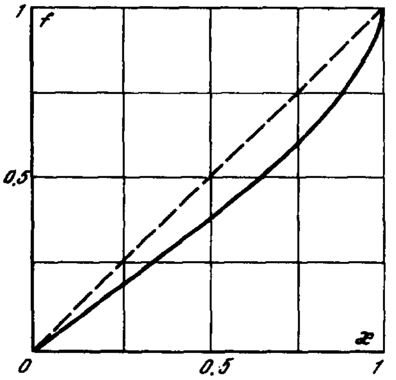

Fig. 1.

$$F = -2\pi\sigma_1\sigma_2 + 8\pi(\mathbf{i}_1 \cdot \mathbf{i}_2)/c^2 \tag{2.1}$$

If we assume that the discs have a surface density ρ_A and consist of dust, for which the trace of the energy–momentum tensor is the scalar density (multiplied by c^2), we have

$$\sigma_A = \rho_A\gamma_A^2(1+\upsilon_A/c^2), \quad \mathbf{i}_A = \rho_A\gamma_A^2\boldsymbol{\upsilon}_A$$

and the velocity is directed along the tangent to the concentric circles. We then obtain from the expression for the force density

$$F = -2\pi\sigma_1\sigma_2\left(1 - \frac{2\upsilon_1/c}{1+\upsilon_1^2/c^2}\frac{2\upsilon_2/c}{1+\upsilon_2^2/c^2}\right) \tag{2.2}$$

Hence, as in the case of dust-like rings, in dust-like extended discs attraction in the internal regions predominates over repulsion. Unlike rings, for sufficiently close spacings between the discs the expression for the force density approaches a constant (2.2), and hence the dimensions of discs that are in equilibrium cannot appreciably exceed the distance between them. In this case a more detailed investigation is necessary, which will be carried out in Section 3 to calculate the projection of the principal force vector on the axis of symmetry for Kerr discs.

The projection of the principal force vector, acting on a plane disc in an external field, on the axis of symmetry is given by the following expression (the square brackets denote discontinuities in the corresponding quantities in the plane of the disc)

$$\frac{1}{4\pi G}\int\left[\frac{\partial\phi}{\partial n}\right]\frac{\partial\phi}{\partial n}dS + \frac{1}{16\pi G}\int[\mathbf{H}]\mathbf{H}dS = 0 \tag{2.3}$$

It was assumed in (2.3) that the discontinuity in the gravimagnetic field $\mathbf{H}$ only has a tangential component. Outside sources the gravimagnetic field $\mathbf{H}$, by (1.3), is a potential field, i.e. $\mathbf{H} = 2\nabla\psi$. In the axisymmetric case the functions ϕ and ψ arise in the Newtonian limit for the Ernst function

$$E \approx 1 + 2(\phi + i\psi)/c^2$$

We will further assume that two axisymmetric discs rotating around a common axis of symmetry z at a distance l from one another act as sources of steady gravitational field. By (2.3) the necessary condition of equilibrium of, say, disc 2 in the gravitational field of disc 1 is the following

$$\frac{1}{2G}\int\left[\left[\frac{\partial\phi_2}{\partial n}\right]\frac{\partial\phi_1}{\partial z} + \left[\frac{\partial\phi_2}{\partial\tau}\right]\frac{\partial\phi_1}{\partial\rho}\right]\rho d\rho = 0 \tag{2.4}$$

In the region of the edge of the disc the fields ϕ_2 and ψ_2 are singular, and hence in (2.4) a correct passage to the limit from the closed smooth surface to the surface of the disc is necessary.

3. We will choose as a specific model of each of the discs the weak-field approximation for the Kerr solution in which the radii of the discs are much greater than their corresponding gravitational radii Gm_k/c^2. The overall moments of momentum of discs 1 and 2 are given by the formulae $K_1 = m_1c\alpha$, $K_2 = m_2c\beta$, respectively. These discs can be treated as cracks in space–time in which a transition to another copy of space–time occurs (similar to the transition from one sheet of a Riemann surface to another in the theory of multivalued analytic functions of a complex variable). Thus, we will assume that the potentials of the discs are given by the following expressions

$$\phi_a = -G\frac{m_a}{2}\left(\frac{1}{r_{+a}} + \frac{1}{r_{-a}}\right), \quad \psi_a = iG\frac{m_a}{2}\left(\frac{1}{r_{+a}} - \frac{1}{r_{-a}}\right), \quad a = 1,2 \tag{3.1}$$

$$r_{\pm a} = \sqrt{(z - l_a \mp iR_a)^2 + \rho^2}$$

where R_a is the radius of the ath disc and l_a is the coordinate of its position on the z axis. We will further put $l_1 = l$, $l_2 = 0$, $R_1 = \alpha$, $R_2 = \beta$. In Fig. 2 we show the pattern of lines of equal potential $\phi_2 = \text{const}$ (where const $= \operatorname{tg}(\pi \times 0.025n)$ and n is an integer). The pattern has two axes of symmetry—horizontal $z = 0$ and vertical $\rho = 0$, and hence we only show the quadrant $z \geqslant 0$, $\rho \geqslant 0$ here. The point A in Fig. 2 is an unstable position of equilibrium of a saddle type. In the region of the edge of the disc there is a singular increase in the gravitational forces similar to the concentration of stresses in the region of the edges of a crack with a similar asymptotic form. In fact, by (3.1) the complex potential in the region of the edge of the disc has the asymptotic form

$$\phi_1 + i\psi_1 \approx -\frac{m_1 G}{\sqrt{2\alpha(\xi-\alpha)}}, \quad \xi \equiv \rho + iz \tag{3.2}$$

It follows from the form of the potentials $\phi_a(\psi_a)$ that they are symmetrical (antisymmetrical) about the planes of the corresponding discs. Hence, the condition of equilibrium (2.4) can be rewritten in the form

$$\int \frac{\partial\phi_1}{\partial n}\frac{\partial\phi_2}{\partial z}\rho d\rho = \int \psi_2 \frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_1}{\partial\rho}\right)d\rho \tag{3.3}$$

In the integral on the right-hand side of (3.3) $\lim \psi_2 = (\beta^2-\rho^2)^{-1/2}$ as $z \to 0$. If we change to the variable $t = \beta^2 - \rho^2$, we obtain the right-hand side of (3.3)

$$2m_1 m_2 G \int\limits_0^{\beta^2} \frac{dt}{\sqrt{t}} \frac{\partial(\beta^2-t)}{\partial t}\frac{\partial\psi}{\partial t}dt \tag{3.4}$$

Here

$$\psi = l\sqrt{2}/\sqrt{AB}, \quad A = \sqrt{-b+\sqrt{B}}, \quad B \equiv b^2 + 4\alpha^2 l^2, \quad b = t + \alpha^2 - \beta^2 + l^2$$

The limit must be taken more carefully on the left-hand side of (3.3). To calculate the left-hand side of (3.3) we will consider taking the limit as $\varepsilon \to 0$ of the integral over the surface consisting of a circle $0 \leqslant \rho \leqslant \beta$, $z = \varepsilon/(2\beta) > 0$ and part of a torus

$$z^2 + (\rho-\beta)^2 = \varepsilon^2/(2\beta)^2 \tag{3.5}$$

with $0 \leqslant \rho - \beta \leqslant \varepsilon/(2\beta)$, $z > 0$. We will put

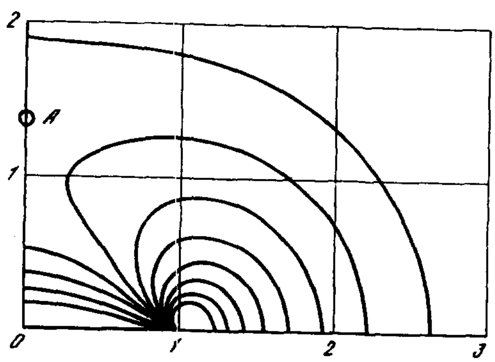

Fig. 2.

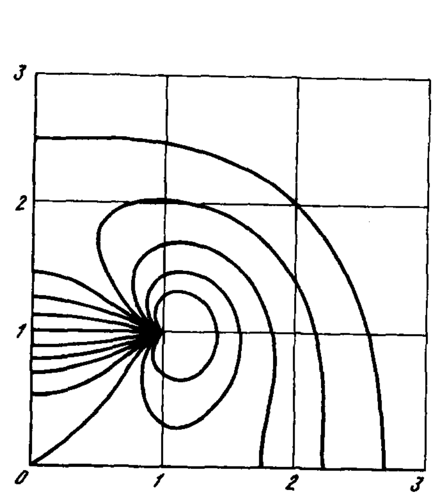

Fig. 3.

$$f(t)=(2AB)^{-1/2}B^{-1}(l(l^2-t)(l^2-t+4\alpha^2)-4\alpha^2l^2+\sqrt{B}(l^2-t+2\alpha^2))$$

The integral on the left-hand side of (3.3) can then be represented in the form of the sum of integrals over the surfaces indicated above

$$\frac{Gm_1m_2\beta}{2\sqrt{2}}\left(\varepsilon\int_0^{\beta^2} f(t+\alpha^2-\beta^2)F(t)dt-\frac{2f(\alpha^2-\beta^2)}{\sqrt{\varepsilon}}\right) \tag{3.6}$$

$$F(t)=\frac{2t-\hat{t}}{\hat{t}^3(\hat{t}-t)^{1/2}},\quad \hat{t}\equiv\sqrt{t^2+\varepsilon^2}$$

The last term in the square brackets corresponds to the integral over a quarter of the circle (3.5), obtained using the asymptotic form (3.2). In order to take the limit we will first replace the factor $2/\sqrt{\varepsilon}$ in this term by the identically equal expression

$$\frac{2}{\sqrt{\varepsilon}}=\varepsilon\int_0^{\infty} F(t)dt \tag{3.7}$$

Splitting the interval of integration in (3.7) into two parts (from 0 to β^2 and from β^2 to ∞), after substituting into (3.6) and taking the limit as $\varepsilon\to 0$, we obtain, using (3.4) and (3.3), the required equation for determining the distance between two discs l as a function of their radii α and β

$$-f(\alpha^2)+\beta\int_0^{\beta^2} f'(t+\alpha^2-\beta^2)\frac{dt}{\sqrt{t}}-2\int_0^{\beta^2}((\beta^2-t)\psi'(t))'\frac{dt}{\sqrt{t}}=0 \tag{3.8}$$

This has the simple solution

$$l=\alpha+\beta \tag{3.9}$$

We will expand the left-hand side of (3.8) in a series of powers of β and take into account terms up to β^3 inclusive. We obtain

$$-\frac{l^2-\alpha^2}{(l^2+\alpha^2)^2}+\frac{2\alpha\beta(3l^2-\alpha^2)}{(l^2+\alpha^2)^3}+\frac{3\beta^2(l^4-6l^2\alpha^2+\alpha^4)}{(l^2+\alpha^2)^2}-4\alpha\beta^3\frac{(5l^4-10l^2\alpha^2+\alpha^4)}{(l^2+\alpha^2)^5}+\ldots=0 \tag{3.10}$$

It follows from (3.10) that $\beta=l-\alpha+o((l-\alpha)^3)$. Numerical solution of Eq. (3.8) shows that the solution (3.9) is exact (for a numerical investigation it is best to choose a system of units in which $\alpha=1$). Using the expressions for α and β in terms of the moments of momentum of the discs we obtain

$$cl=K_1/m_1+K_2/m_2 \tag{3.11}$$

Expression (3.11) gives an upper limit for the distance between actual discs which can be in equilibrium when the Newtonian attraction is compensated by gravimagnetic repulsion. In Fig. 3 we show the pattern of lines of equal potential for two discs in equilibrium for the same values of the potentials as in Fig. 2. The pattern has two axes of symmetry—horizontal $z=0$ and vertical $\rho=0$, and hence we only show the quadrant $z\geqslant 0$, $\rho\geqslant 0$.

4. We will now consider the condition of equilibrium between a disc described by solution (3.1) and a compact source on the axis of rotation z with mass m_1 and rotational moment $K_2=c\beta m_2$ directed along the z axis (such Newtonian analogues arise for Curzon solutions with rotation in the General Theory of Relativity). Then, equating the sum of the Coulomb and gravimagnetic forces, calculated using (1.2), to zero, we obtain a condition corresponding to the case when the sum of the first two terms on the left-hand side of (3.10) vanishes.

As $\beta\to 0$ this condition of equilibrium reduces asymptotically to condition (3.10). Note that this position of equilibrium is unstable with respect to lateral displacements from the z axis when $\alpha\leqslant l\leqslant\alpha\sqrt{3}$.

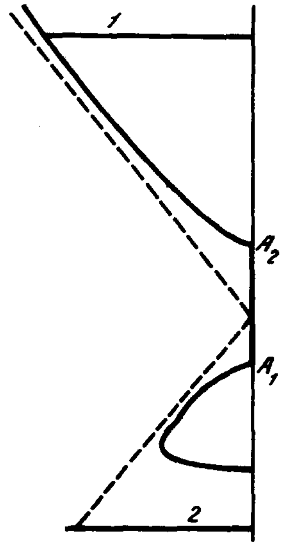

Fig. 4.

5. The results obtained above are a Newtonian interpretation of the exact solutions with a regular behaviour of space–time along the section of the axis of symmetry between the rotating bodies [10], obtained here within the framework of the General Theory of Relativity using the method employed in [11]. These results are in a sense a summary of those obtained in [5, 6, 7, 9].

In Fig. 4 we show attracting rotating discs in equilibrium, where the radius of disc 2 is fixed while discs 1 have radii bounded by the dashed curves in Newtonian theory and by the continuous curves in the General Theory of Relativity. The pattern has an axis of symmetry which coincides with the axis of the discs, and hence we have shown only half the pattern here. Along section A_1A_2 the second attracting body is converted into a black hole, since along this part the condition for the existence of a horizon is satisfied. Hence, equilibrium is possible between discs and black holes, where the black holes can only appear in the neighbourhood of the saddle point A. We recall that in the Weil coordinates employed a black hole is a section of the axis of symmetry.

There is another curious result that can only be obtained in the accurate theory from Einstein's equations: if two discs with equal masses and equal moments of momentum converge without limit, an extremal Kerr solution is obtained with double mass and rotation parameter equal to the mass multiplied by G/c^2. In the General Theory of Relativity there is one other free parameter to describe the equilibrium configurations in addition to the distance between the discs. To obtain the values of these two parameters the conditions for there to be no conical points and closed timelike world lines are used.

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